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Miscellaneous Properties on Equi-Eccentric Graphs

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1. Introduction

We deal with only connected graphs throughout this paper. The eccentricity $e(v)$ of a vertex v of a connected graph G is the number $\max_{u \in V(G)} d(u,v)$, where $d(u,v)$ stands for the distance between u and v. A central vertex of a connected graph G is a vertex v with the property that the maximum possible distance between v and any other vertex is as small as possible, this distance being called the radius, denoted by $r(G)$, that is, $r(G) = \min_{v} \max_{w} d(v,w)$. The subgraph induced by the set of central vertices of G is called the center of G. Then a graph G is r-equi-eccentric (or briefly, r-equi) if $e(v) = r(G)$ for every vertex of G, that is, a graph whose center is itself. An r-equi-eccentric graph G is said r-minimal if $G - e$ is no longer r-equi for any edge e of G. An r-equi-eccentric graph G of order p is r-minimum if G has the least number of edges among all r-equi-eccentric graphs of order p. We denote by $N(v)$ the neighborhood of a vertex v of G consisting of the vertices of G adjacent with v. The closed neighborhood $N[v]$ of v is defined as $N[v] = N(v) \cup \{v\}$.

All other definitions and notations used in this paper can be found in [1] or [2].

We first present a few fundamental properties on equi-eccentric graphs.

Proposition 1.1. Every equi-eccentric graph G except K_2 is a block.

Proof. Every vertex of G is a central vertex by the definition and the center of every connected graphs lies in its single block. □

Proposition 1.2. Let G be r-equi of order p with maximum degree Δ, then the following inequality holds:

$$\Delta \leqq p - 2(r - 1).$$

Proof. Let v be an arbitrary vertex of G and u be a vertex with $d(u,v) = r$. By Proposition 1.1, G is a block or K_2. If G is K_2 then the theorem is true. On the other hand, if G is a block there is at least one cycle containing both u and v. By C we denote the smallest one among those cycles. Then note that $|V(C)| \geqq 2r$ since $d(u,v) = r$, and $|V(C) \cap N[v]| = 3$ since C is the smallest such cycle. Thus the following inequalities hold:

$$|V(G)| - |N[v]| \geqq |V(C)| - |V(C) \cap N[v]| \geqq 2r - 3.$$

Since $|V(G)| - |N[v]| = p - (\deg v+1)$, we have

$$\deg v \leqq p - 2(r - 1) \text{ for every vertex } v \text{ of } G,$$

completing the proof. □

2. Operations producing equi-eccentric graphs

In this section, we exhibit several interesting operations to produce equi-eccentric graphs. We omit proofs when they are immediate from the constructions.

(I) Mycielski's operation

Generating Mycielski's operation to an arbitrary graph $G = (V,E)$ with p vertices and q edges, we define its (Mycielski) successor $\hat{G} = (\hat{V},\hat{E})$ as follows:

(i) For each $x \in V$, generate its twin x', call the set of twins V'.

(ii) Join x' to $N(x)$ in G, for every $x' \in V'$.

(iii) Create a new vertex z and join it to all twin vertices $x' \in V'$.

Example 1. Let G be the graph $K_4 - e$. Then its Mycielski successor $\hat{G}$ is as follows; see Figure 2.1.

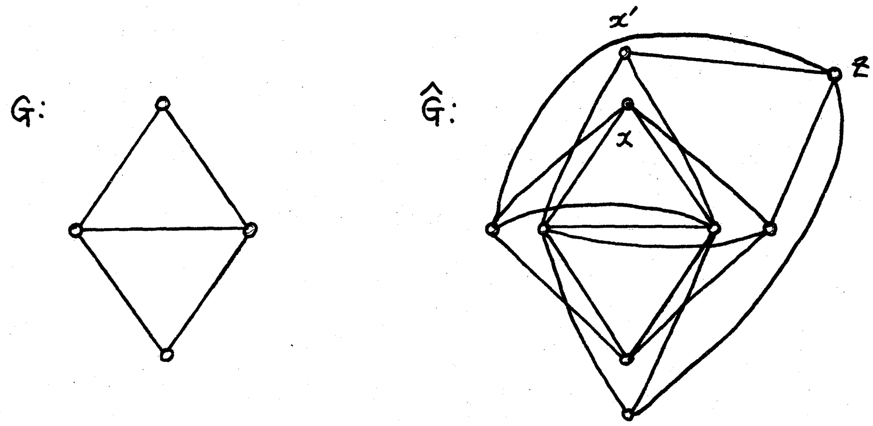


Figure 2.1.

Let G be a (p,q)-graph, then $\hat{G}$ is a $(2p+1, 3q+p)$-graph.

Note that a graph with $p+1$ vertices is 2-equi if it contains no $K(1,p)$ and the $\max_{u,v \in V(G)} d(u,v) = 2$.

Using the same notations above, we prove the following result.

Theorem 2.1. If G is 2-equi then $\hat{G}$ is 2-equi.

Proof. It is immediate from the construction that $\hat{G}$ does not contain $K(1,2p)$. We verify the second condition. Let $\hat{d}(u,v)$ denote distances in $\hat{G}$.

(i) $u, v \in V$: $\hat{d}(u,v) = d(u,v)$, provided that $d(u,v) \leq 2$

(ii) $u', v' \in V'$: $\hat{d}(u',v') \leq d(u',z) + d(v',z) = 2$

(iii) $u \in V$, z: $\hat{d}(u,z) = 1 + d(v',z) = 2$, where v is a neighbor of u in G.

(iv) $u' \in V'$, z: $\hat{d}(u',z) = 1$, by construction.

(v) $u \in V$, $v' \in V'$: If $d(u,v) = 1$ then $\hat{d}(u,v') = 1$. Otherwise let w be a common neighbor of u and v. There exists such a vertex w because G is 2-equi.
Then $\hat{d}(u,v') = d(u,w) + \hat{d}(w,v') = 2$.

Thus $\hat{G}$ is 2-equi. □

(II) The join operation

Theorem 2.2. If G is 2-equi, then $G + \overline{K}_n$ ($n \geq 2$) is 2-equi. □
(see Figure 2.2).

$C_4 + \overline{K}_2$:

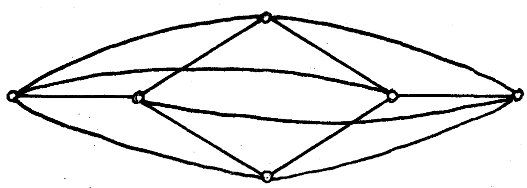


Figure 2.2.

(III) Operations to produce the minimal 2-equi-eccentric graphs

The corona $G_1 \circ G_2$ of two graphs G_1, G_2 with order p_1 and p_2 is defined as the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i-th vertex of G_1 to each vertex in the i-th copy of G_2. In Figure 2.3, we illustrate $C_4 \circ K_2$.

$C_4 \circ K_2$:

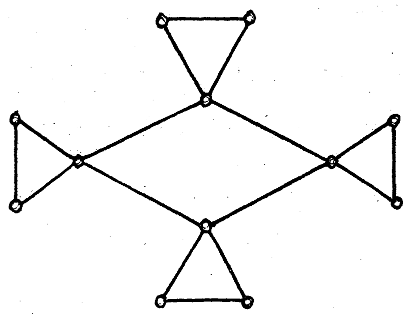


Figure 2.3.

We define the graph $G_n = K_n \circ K_1 + K_1$ $(n \geqq 2)$ as the graph obtained from $K_n \circ K_1$ by adding a new vertex z and joining z to the vertices of degree 1 of $K_n \circ K_1$. In Figure 2.4, we illustrate the graph $K_3 \circ K_1 + K_1$.

Theorem 2.3. The graph $G_n = K_n \circ K_1 + K_1$ $(n \geqq 2)$ obtained by the operation above is minimal 2-equi. □

$K_3 \circ K_1 + K_1$:

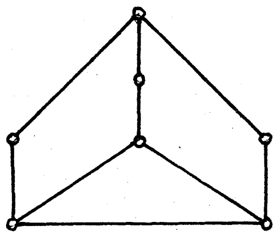


Figure 2.4.

(IV) The cartesian product operation

All of the three operations mentioned above produce 2-equi-eccentric graphs, we now present other operations to produce r-equi-eccentric graphs for an arbitrary integer $r \geq 2$.

The <u>cartesian product</u> $G = G_1 \times G_2$ has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1,u_2) and (v_1,v_2) of G are adjacent if and only if either

(1) $u_1 = v_1$ and $u_2v_2 \in E(G_2)$

or

(2) $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.

<u>Theorem 2.4.</u> Let G_1, G_2 be r_1, r_2-equi, then their cartesian product $G = G_1 \times G_2$ is $(r_1 + r_2)$-equi. □

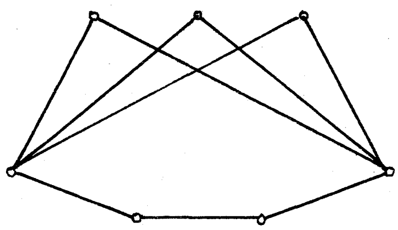
As an immediate consequence of Theorem 2.4, we obtain the next result.

<u>Corollary 2.4.1.</u> The r-cube $Q_r = (K_2)^r$ is r-equi. □

(V) <u>The shift operation by P_n</u>

Let F be any given graph, then define a graph $G_r(F)$ $(r \geq 2)$ consisting of F, a copy of P_{2r} and all edges joining two end-vertices of P_{2r} to the vertices of F. Figure 2.5 illustrates the graph $G_2(\overline{K}_3)$.

$G_2(\overline{K}_3)$:



<u>Figure 2.5.</u>

Theorem 2.5. Let F be an arbitrary graph, then the graph $G_r(F)$ $(r \geqq 2)$ is r-equi. □

Corollary 2.5.1. For any given nonempty graph F and an integer r, there exists an r-equi-eccentric graph containing F as an induced subgraph. □

Note that the above corollary suggests that it is impossible to characterize r-equi-eccentric graphs in terms of forbidden subgraphs.

3. 2-equi-eccentric graphs

We denote the degree of a vertex v_i by d_i for the sake of convenience.

Proposition 3.1. There are no 2-equi-eccentric graphs G with minimum degree $\delta = 3$ and $q \leqq 2p - 5$, other than the Petersen graph.

Proof. We show that G is isomorphic to the Petersen graph, if G is 2-equi with $\delta = 3$ and $q \leqq 2p - 5$. Let v_i be a vertex of degree 3. By v_2, v_3, v_4 we denote vertices adjacent to v_1, and the (p - 4) remaining vertices in G by $v_5, v_6, \ldots, v_p$. Each vertex v_i $(5 \leqq i \leqq p)$ is adjacent to at least one vertex of v_2, v_3 and v_4, otherwise $d(v_i, v_1) \geqq 3$.
From this fact the inequality (1) follows:

(1) $$d_2 + d_3 + d_4 \geqq p - 1.$$

On the other hand, the inverse inequality of (1) follows from the facts that $\sum_{i=1}^{p} d_i = 2q \leqq 4p - 10$ and $\sum_{i=5}^{p} d_i \geqq 3(p - 4)$ since $d_i \geqq \delta = 3$:

(2) $$d_2 + d_3 + d_4 \leqq p - 1$$

Thus we obtain the following equalities (3) and (4):

(3) $d_2 + d_3 + d_4 = p - 1$

(4) $d_i = 3 \quad (5 \leqq i \leqq p)$

From (3) it follows at once that

$$N(v_i) \cap N(v_j) = \{v_1\} \quad (i \neq j,\ 2 \leqq i,j \leqq 4)$$

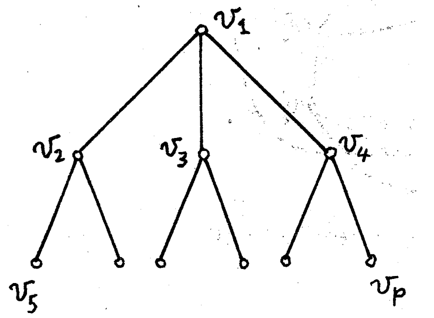


Figure 3.1. A stage of the proof of Proposition 3.1

Applying the same argument for each vertex v_i $(5 \leqq i \leqq p)$ instead of v_1 since $d_i = 3$ from (4), then we see that $d_i = 3$ for i $(2 \leqq i \leqq 4)$ and so G is cubic. Furthermore denoting by V_i' the vertex set $N(v_i) - \{v_1\}$ for i = 2, 3 and 4, we have that $|V_i'| = 2$. Without loss of generality we may assume that $V_2' = \{v_5, v_6\}$, $V_3' = \{v_7, v_8\}$ and $V_4' = \{v_9, v_{10}\}$. On a basis of the fact that G is 2-equi, we see that the graph $G' = G - \{v_1,v_2,v_3,v_4\}$ is connected, which implies that G' is a 6-cycle. Thus it is easy to verify that the graph with the properties mentioned above is isomorphic to the Petersen graph, see Figure 3.2. □

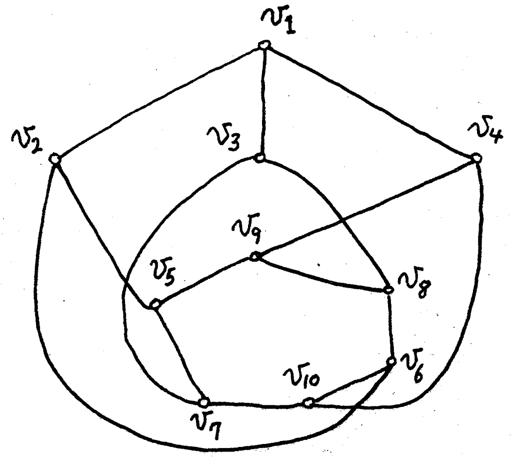


Figure 3.2. The Petersen graph

Theorem 3.1. If a (p,q)graph G is 2-equi, then $q \geqq 2p - 5$.

Proof. Let G be 2-equi then G is a block by Proposition 1.1. Thus $\delta(G) \geqq 2$. If $\delta(G) \geqq 4$ the theorem is true since $q \geqq 2p$. If $\delta(G) = 3$ then it follows from Proposition 3.1 that $q \geqq 2p - 5$. We may thus assume that $\delta = 2$. Let v be a vertex of degree 2 and u, w be vertices adjacent to v in G. We define three vertex sets I, U, W, see Figure 3.3, and denote their cardinality by i, j, k respectively.

$$I = N(u) \cap N(w) - \{v\}.$$

$$U = N(u) - I - \{v\}.$$

$$W = N(w) - I - \{v\}.$$

Since $d(x,y) \leqq 2$ for any pair of verices $x \in U$, $y \in W$, x is connected to y in the induced subgraph $G' = \langle G - \{v,u,w\}\rangle$. Thus the induced graph $G'' = \langle U \cup W\rangle$ is in a connected component of G', which implies that G'' has at least $j + k - 1$ edges. Therefore, we obtain the inequality as required, since $i + j + k = p - 3$.

$$q \geqq 2 + j + 2i + k + (j + k - 1) = 2(i + j + k) + 1$$
$$= 2p - 5 \qquad \square$$

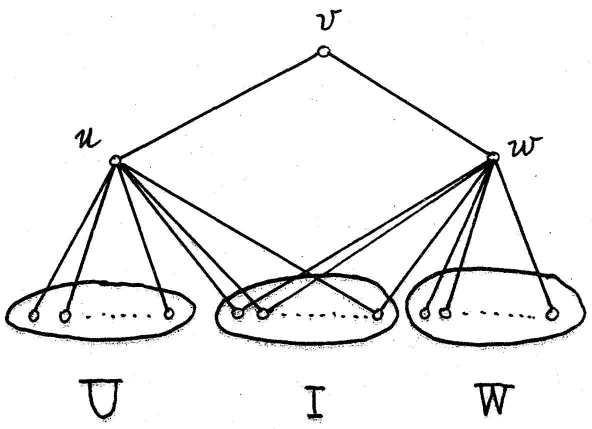


Figure 3.3.

Before presenting the characterization theorem for the minimum 2-equi-eccentric graphs, we require a definition.

For any tree T, we denote by T' the subtree obtained on deleting the endvertices of T. Then a double star is a tree T such that $T' = K_2$: it is denoted by T(m,n) when m endvertices are adjacent to one vertex of this K_2 and n to the other.

Lemma 3.1. Let T be a tree. If there is a partition {U,W} of V(T) such that

(1) $d(u,w) \leq 2$ for any $u \in U$ and $w \in W$,

(2) both U and W are dominating sets of T.

Then T is either a star or a double star.

Proof. It is easy to see that if T is either a star or a double star then there is such a partition (see Figure 3.4).

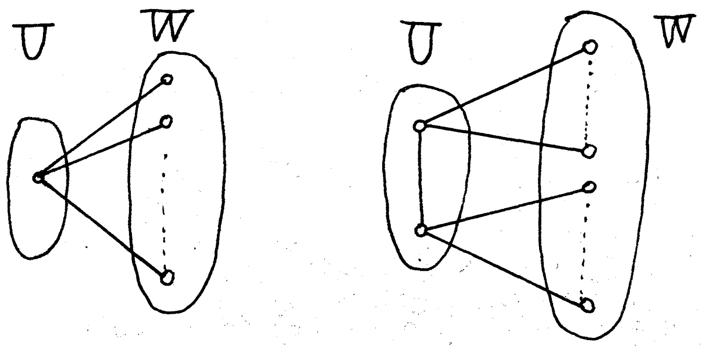


Figure 3.4. A star and a double star

Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$. Since T is acyclic it follows from the condition (1) that T cannot contain P_4, $2P_3$ and $3P_2$ of the form in Figure 3.5.

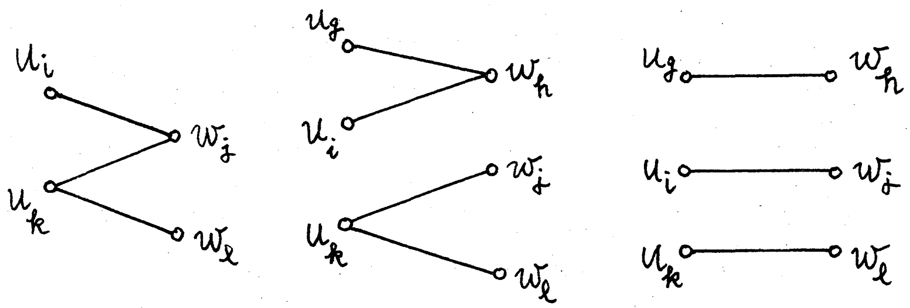


Figure 3.5. Forbidden subgraphs P_4, $2P_3$ and $3P_2$.

We call them the forbidden subgraphs P_4, 2P3 and $3P_2$ for T. Without loss of generality we may assume that $|U| \leqq |W|$.

Let $W(u_i) = N(u_i) \cap W$ for $1 \leqq i \leqq m$. We first note that no sets $W(u_i)$ are empty. Since otherwise $u_i \notin N(w)$ for any $w \in W$, contradicting (2) of the lemma.

Using the assumption that $|U| \leqq |W|$, we show that $W(u_i) \cap W(u_j) = \phi$ for any i, $j(i \neq j)$. Suppose that $W(u_i) \cap W(u_j) \neq \phi$ for some i and $j(i \neq j)$ then $W(u_i) = W(u_j)$ since otherwise T would contain the forbidden subgraph P_4. Furthermore $W(u_i)$ $(=W(u_j))$

consists of only a single vertex, otherwise T would contain the cycle C_4, contradicting T a tree. Then since $|U| \leq |W|$, there is a vertex u_k in U such that $|W(u_k)| \geq 2$. This implies that T contains the forbidden subgraph $2P_3$ (see Figure 3.6).

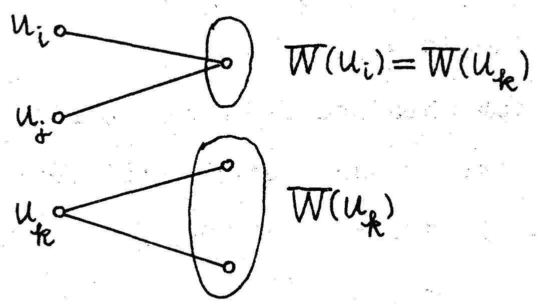


Figure 3.6.

Therefore $W(u_i) \cap W(u_j) = \phi$ for any i and j ($i \neq j$). Consequently if $|U| \geq 3$ then T would contain the forbidden subgraph $3P_2$. Finally we get that $|U| = 1$ or $|U| = 2$. And it is easy to see that T is either a star or a double star depending on whether $|U| = 1$ or 2. $\square$

In the following theorem we use the next terminology.

The graph $K_3(\ell,m,n)$ is the graph obtained from K_3 adding ℓ, m, n pendent edges from each vertex of K_3, respectively, Figure 3.7 illustrates the graph $K_3(1,2,3)$.

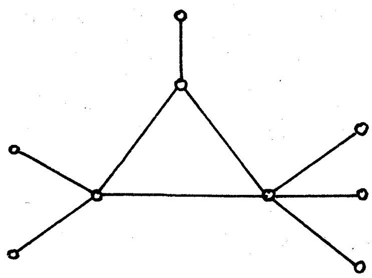


Figure 3.7. $K_3(1,2,3)$

Theorem 3.2. Let G be a minimum 2-equi-eccentric graph other than the Petersen graph, then G is one of the followings:

(I) The graph obtained from the double star T(m,n) by adding a new vertex v and joining v to every vertex of degree 1 of T(m,n), where m, n are arbitrary positive integers, see Figure 3.8(a).

(II) The graph obtained from $K_3(\ell,m,n)$, ℓ, m, n $\geqq$ 1, by adding a new vertex v and joing v to every vertex of degree 1 of $K_3(\ell,m,n)$, see Figure 3.8(b).

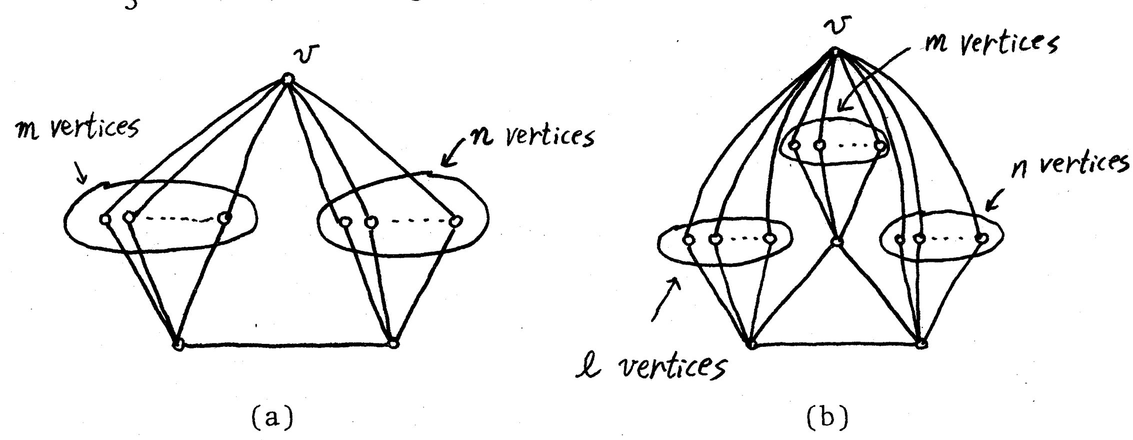


Figure 3.8.

Proof. If G is a minimum 2-equi-eccentric graph other than the Petersen graph, then the minimum degree δ of G is 2 by Theorem 3.1. Let v be a vertex of degree 2 in G and u, w be the vertices adjacent to v. Then every vertex of V(G) - {u,v} is adjacent to either u or w, since G is 2-equi. Set three vertex-subsets I, U, W as follows:

$$I = N(u) \cap N(w)$$

$$U = N(u) - I$$

$$W = N(w) - I$$

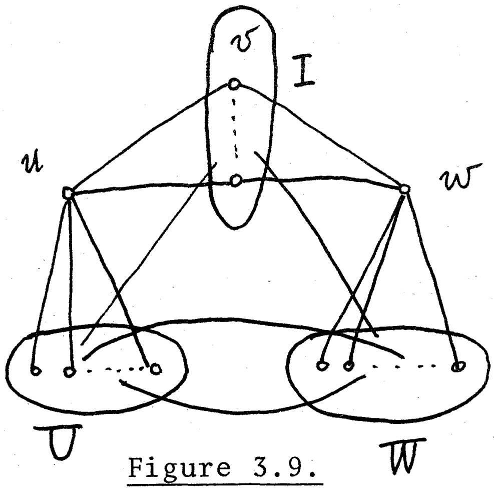


Figure 3.9.

Let $|I| = p_1$, $|U| = p_2$ and $|W| = p_3$, then neither p_2 nor p_3 is 0. Because if both p_2 and p_3 are 0 then

$$q = 2p_1 > 2p_1 - 1$$
$$= 2p - 5, \quad \text{contradicting to the hypothesis}$$

that G is minimum 2-equi. If one of U or W is empty and the other is not, then u or w would be a cutvertex of G contradicting to the fact that G is 2-equi by Proposition 1.1.

Set $G' = G - \{u,v\}$

$$= \langle I \cup U \cup W\rangle_G$$

and

$$T = \langle U \cup W\rangle_G$$
$$= \langle U \cup W\rangle_{G'}.$$

Then since G is 2-equi, $d'(x,y) \leqq 2$ for any $x \in U$ and $y \in W$, where $d'(x,y)$ stands for the distance between x and y in G' (see Figure 3.9). So T lies in a connected component H of G'. On the other hand, we have

$$q(H) \leqq q' \leqq q - (\deg u + \deg w)$$
$$\leqq 2p - 5 - (2p_1 + p_2 + p_3)$$
$$= p_2 + p_3 - 1.$$

So $p(H) \leqq p_2 + p_3$ since H is connected. The fact that $H \cong T$ follows immediately from the inequality $p(H) \leqq p_2 + p_3 = p(T)$ and that H T. Thus we obtain the following facts:

(i) $T = \langle U \cup W \rangle_G$ is a tree

(ii) the vertices u and w are not adjacent in G

(iii) $\langle I \rangle = G' - T$ is totally disconnected.

It follows from that $d(u,y) = 2$ and the condition (ii) that $N_G(y) \cap U = N_T(y) \cap U \neq \phi$, for any $y \in W$. Similarly, we obtain that $N_T(x) \cap W \neq \phi$ for any $x \in U$. We thus obtain

(iv) both U and W are dominating sets of $T = \langle U \cup W \rangle_G$.

Applying Lemma 3.1 and (iv), we have that T is either a star or a double star. Then we obtain the graphs illustrated in Figure 3.8(a), (b) according to T is a star or a double star. □

References

[1] M. Behzad, G. Chartrand and L. Lesniak-Foster, Graphs and digraphs, Prindle, Weber & Schmidt, Reading(1979)

[2] F. Harary, Graph Theory, Addison-Wesley, Reading(1969)